

INFINITESIMAL INDEX: COHOMOLOGY COMPUTATIONS

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Dedicated to Tonny Springer

ABSTRACT. In this note we study the equivariant cohomology with compact supports of the zeroes of the moment map for the cotangent bundle of a linear representation of a torus and some of its notable subsets, using the theory of the infinitesimal index, developed in [8]. We show that, in analogy to the case of equivariant K -theory dealt with in [7] using the index of transversally elliptic operators, we obtain isomorphisms with notable spaces of splines studied in ([2], [3]).

1. INTRODUCTION

Let G be a compact Lie group with Lie algebra \mathfrak{g} , N a manifold with G -action and equipped with a G -equivariant 1-form σ .

From this setting, one has a moment map $\mu^\sigma : N \rightarrow \mathfrak{g}^*$. A particularly important case is that of $N = T^*M$, the cotangent bundle of a manifold with a G action, equipped with the canonical action form. In this case, the zeroes of the moment map is a subspace T_G^*M whose equivariant K -theory is strongly related to the index of transversally elliptic operators as shown in [1].

In order to understand explicit formulas for such an index, in [8] we have introduced the *infinitesimal index* infindex , a map from the equivariant cohomology with compact supports of the zeroes of the moment map to distributions on \mathfrak{g}^* .

We have proved several properties for this map which, at least in the case of the space T_G^*M , in principle allow us to reduce the computations to the case in which G is a torus and the manifold is a complex linear representation of G . A finite dimensional complex representation of a torus is the direct sum of one dimensional representations given by characters. If X is a list of characters, we denote by M_X the corresponding linear representation which is naturally filtered by open sets $M_{X, \geq i}$ where the dimension of the orbit is $\geq i$.

In this paper, we first compute the equivariant cohomology of the open sets $M_{X, \geq i}$, and also of some slightly more general open sets in M_X . This part of our paper, namely Sections 2 and 3, does not use the notion of infinitesimal index. The results are obtained from the structure of the algebra $S[\mathfrak{g}^*][(\prod_{a \in X} a)^{-1}]$ as a module over the Weyl algebra studied in [5].

In Section 4 we apply the results we have obtained to the equivariant cohomology of the open set M_X^{fin} of points with finite stabilizer. Using Poincaré duality, we remark that the equivariant cohomology with compact supports $H_{G,c}^*(T_G^*M_X^{fin})$ of $T_G^*M_X^{fin}$ is isomorphic as a $S[\mathfrak{g}^*]$ -module to a remarkable finite dimensional space $D(X)$ of polynomial functions on \mathfrak{g}^* , where $S[\mathfrak{g}^*]$ acts by differentiation. The space $D(X)$ is defined as the space of solutions of a set of linear partial differential equations combinatorially associated to X and has been of importance in approximation theory (see for example [2], [3]).

At this point the notion of infinitesimal index comes into play. We show in Theorem 5.12 that the infinitesimal index gives an isomorphism between $H_{G,c}^*(T_G^*M_X^{fin})$ and $D(X)$. After this, we show that, for each i , the infinitesimal index establishes an isomorphism between $H_{G,c}^*(T_G^*M_{X,\geq i})$ and a space of *splines* $\tilde{\mathcal{G}}_i(X)$, introduced in [6], (cf. (17)) and generalizing $D(X)$.

It should be mentioned that, in the previous paper [7], similar results have been proved, using the index of transversally elliptic differential operators, in order to compute the equivariant K-theory of the spaces $T_G^*M_{X,\geq i}$.

This paper represents a sort of “infinitesimal” version of [7] and will be used, in a forthcoming paper [9], to give explicit formulas for the index of transversally elliptic operators.

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2. A SPECIAL MODULE

2.1. A module filtration. Let G be a compact torus with Lie algebra \mathfrak{g} and character group Λ . We are going to consider Λ as a lattice in \mathfrak{g}^* .

We need to recall some general results proved in [5]. Let us fix a list $X = (a_1, \dots, a_m)$ of non zero characters in $\Lambda \subset \mathfrak{g}^*$. Let $S[\mathfrak{g}^*]$ be the symmetric algebra on \mathfrak{g}^* or in other words the algebra of polynomial functions on \mathfrak{g} . For a list Y of vectors, let us set $d_Y := \prod_{a \in Y} a \in S[\mathfrak{g}^*]$.

Definition 2.2. A subspace \underline{s} of \mathfrak{g}^* is called *rational* (relative to X) if $\underline{s} = \langle X \cap \underline{s} \rangle$.

In general if A is a set of vectors we denote by $\langle A \rangle$ the linear span of A .

We shall denote by \mathcal{S}_X the set of rational subspaces and, for a given $0 \leq k \leq s$, by $\mathcal{S}_X(k)$ the set of the rational subspaces of dimension k .

We shall assume always that X spans \mathfrak{g}^* and need to recall that a *cocircuit* in X is a sublist of X of the form $Y := X \setminus H$ where H is a rational hyperplane.

Definition 2.3. We denote by I_X the ideal in $S[\mathfrak{g}^*]$ generated by the elements d_Y 's, as Y runs over the cocircuits.

One knows that I_X defines a scheme V_X supported at 0 and of length $d(X) = \dim(S[\mathfrak{g}^*]/I_X)$ where $d(X)$ equals the number of bases extracted from X , (see [5], Theorem 11.13).

Consider the localized algebra $R_X := S[\mathfrak{g}^*][d_X^{-1}]$, which is the coordinate ring of the complement of the hyperplane arrangement in \mathfrak{g} , defined by the equations $a = 0$, $a \in X$.

This algebra is a cyclic module over the Weyl algebra $W[\mathfrak{g}]$ of differential operators with polynomial coefficients, generated by d_X^{-1} (Theorem 8.22 of [5]).

In [5], we have seen that this $W[\mathfrak{g}]$ -module has a canonical filtration, the *filtration by polar order*, where we put in degree of filtration $\leq k$ all fractions in which the denominator is a product of elements in X spanning a rational subspace of dimension $\leq k$ (we say that k is the polar order on the boundary divisors). We denote this subspace by $R_{X,k}$. One of the important facts (Theorem 8.10 in [5]) is that

Theorem 2.4. *The $W[\mathfrak{g}]$ -module $R_{X,k}/R_{X,k-1}$ is semisimple, its isotypic components are in 1-1 correspondence with the rational subspaces of dimension k and such a isotypic component is generated by the class of $1/d_{X \cap \underline{s}}$.*

Consider the rank 1 free $S[\mathfrak{g}^*]$ submodule $L := d_X^{-1}S[\mathfrak{g}^*]$ in R_X generated by d_X^{-1} . Set $L_k := L \cap R_{X,k}$, that is the intersection of L with the k -filtration. We obtain for each k an ideal I_k of $S[\mathfrak{g}^*]$ defined by

$$I_k := L_k d_X.$$

For a given rational subspace \underline{s} of dimension k , denote by $I_{\underline{s}} := S[\mathfrak{g}^*]d_{X \setminus \underline{s}}$ the principal ideal generated by $d_{X \setminus \underline{s}}$. Notice that

$$I_{\underline{s}}L = d_{X \cap \underline{s}}^{-1}S[\mathfrak{g}^*] \subset L_k.$$

Thus $I_{\underline{s}} \subset I_k$ and indeed from Theorem 11.29 of [5] one gets

$$I_k = \sum_{\underline{s} \in \mathcal{S}_X(k)} I_{\underline{s}}.$$

If $Q \subset \mathcal{S}_X$ is a set of rational subspaces, we set

$$I_Q = \sum_{\underline{s} \in Q} I_{\underline{s}}$$

for the ideal generated by the elements $d_{X \setminus \underline{s}}$ for $\underline{s} \in Q$.

Associated to \underline{s} , we also consider the list $X \cap \underline{s}$ consisting of those elements of X lying in \underline{s} and we may consider the ideal $I_{X \cap \underline{s}} \subset S[\underline{s}]$, as defined in 2.3, and its extension $J_{X \cap \underline{s}} := I_{X \cap \underline{s}}S[\mathfrak{g}^*]$. Since $S[\mathfrak{g}^*]$ is a free $S[\underline{s}]$ module, the obvious map

$$S[\mathfrak{g}^*] \otimes_{S[\underline{s}]} I_{X \cap \underline{s}} \rightarrow J_{X \cap \underline{s}}$$

is an isomorphism and

$$(1) \quad S[\mathfrak{g}^*]/J_{X \cap \underline{s}} \simeq S[\mathfrak{g}^*] \otimes_{S[\underline{s}]} (S[\underline{s}]/I_{X \cap \underline{s}}).$$

Lemma 2.5. *If \underline{s} is of dimension k , we have that $d_{X \cap \underline{s}}^{-1} S[\mathfrak{g}^*] \subset L_k$ and*

$$(2) \quad d_{X \cap \underline{s}}^{-1} S[\mathfrak{g}^*] \cap L_{k-1} \supset d_{X \cap \underline{s}}^{-1} J_{X \cap \underline{s}}.$$

Proof. We have already remarked the first statement. As for the second, by definition $J_{X \cap \underline{s}}$ is the ideal generated by the elements d_Z where Z is a cocircuit in $X \cap \underline{s}$. This means that Z is contained in $X \cap \underline{s}$ and that $Y := (X \cap \underline{s}) \setminus Z$ spans a subspace of dimension $k-1$. Hence $d_{X \cap \underline{s}}^{-1} d_Z S[\mathfrak{g}^*] = d_Y^{-1} S[\mathfrak{g}^*] \subset L_{k-1}$. \square

Multiplying Formula (2) by d_X , we deduce that

$$(3) \quad I_{\underline{s}} \cap I_{k-1} \supset J_{X \cap \underline{s}} d_{X \setminus \underline{s}} = \sum_{\underline{t} \subset \underline{s}, \underline{t} \in \mathcal{S}_X(k-1)} I_{\underline{t}}.$$

In this way, multiplication by $d_{X \cap \underline{s}}^{-1}$ gives an homomorphism of $S[\mathfrak{g}^*]$ -modules $j_{\underline{s}} : S[\mathfrak{g}^*]/J_{X \cap \underline{s}} \rightarrow L_k/L_{k-1}$ and hence, taking direct sums, a homomorphism $j := \bigoplus_{\underline{s} \in \mathcal{S}_X(k)} j_{\underline{s}}$

$$(4) \quad j : \bigoplus_{\underline{s} \in \mathcal{S}_X(k)} S[\mathfrak{g}^*]/J_{X \cap \underline{s}} \rightarrow L_k/L_{k-1}.$$

We have (Theorem 11.3.15 of [5]):

Theorem 2.6. *The homomorphism j is an isomorphism.*

Using (3), Theorem 2.6 tells us that the summation morphism

$$(5) \quad \tilde{j} : \bigoplus_{\underline{s} \in \mathcal{S}_X(k)} I_{\underline{s}}/J_{X \cap \underline{s}} d_{X \setminus \underline{s}} \rightarrow I_k/I_{k-1}$$

is an isomorphism.

Definition 2.7. A set $Q \subset \mathcal{S}_X$ of rational subspaces is called admissible if, for every $\underline{s} \in Q$, Q also contains all rational subspaces $\underline{t} \subset \underline{s}$.

From Theorem 2.6, we deduce

Proposition 2.8. *1) For any subset $\mathcal{G} \subset \mathcal{S}_X(k)$*

$$(6) \quad \left(\sum_{\underline{s} \in \mathcal{G}} I_{\underline{s}} \right) \cap I_{k-1} = \sum_{\underline{t} \subset \underline{s} \in \mathcal{G}, \underline{t} \in \mathcal{S}_X(k-1)} I_{\underline{t}}.$$

2) Given an admissible subset $Q \subset \mathcal{S}_X$ and a rational subspace $\underline{s} \in Q$ of maximal dimension k , then

$$(7) \quad I_{\underline{s}} \cap I_{Q \setminus \{\underline{s}\}} = I_{\underline{s}} \cap I_{k-1} = \sum_{\underline{t} \subset \underline{s}, \underline{t} \in \mathcal{S}_X(k-1)} I_{\underline{t}}.$$

Proof. 1) By (5), the restriction of \tilde{j} to $\bigoplus_{\underline{s} \in \mathcal{G}} I_{\underline{s}}/J_{X \cap \underline{s}} d_{X \setminus \underline{s}}$ is injective. It follows that

$$\left(\sum_{\underline{s} \in \mathcal{G}} I_{\underline{s}} \right) \cap I_{k-1} = \sum_{\underline{s} \in \mathcal{G}} J_{X \cap \underline{s}} d_{X \setminus \underline{s}} = \sum_{\underline{t} \subset \underline{s} \in \mathcal{G}, \underline{t} \in \mathcal{S}_X(k-1)} I_{\underline{t}}$$

as desired.

2) We first assume that $Q \supset \mathcal{S}_X(k-1)$ so that $Q \setminus \underline{s} = \mathcal{S}_X(k-1) \cup \mathcal{G}$ with $\mathcal{G} \subset \mathcal{S}_X(k)$. If \mathcal{G} is empty, then $I_{Q \setminus \{\underline{s}\}} = I_{k-1}$ and our claim is a special case of 1).

Otherwise $I_{Q \setminus \underline{s}} = I_{k-1} + \sum_{\underline{t} \in \mathcal{G}} I_{\underline{t}}$. Let $b \in I_{\underline{s}} \cap I_{Q \setminus \underline{s}}$. Passing modulo I_{k-1} , we get an element lying in $I_{\underline{s}}/(I_{\underline{s}} \cap I_{k-1})$ and in $(\sum_{\underline{t} \in \mathcal{G}} I_{\underline{t}})/((\sum_{\underline{t} \in \mathcal{G}} I_{\underline{t}}) \cap I_{k-1})$. But the restriction of \tilde{j} to $\oplus_{\underline{t} \in \mathcal{G} \cup \{\underline{s}\}} I_{\underline{t}}/I_{\underline{t}} \cap I_{k-1}$ is injective. It follows that $b \in I_{k-1}$ as desired.

Passing to the general case, set $\tilde{Q} = Q \cup \mathcal{S}_X(k-1)$. We have

$$I_{\underline{s}} \cap I_{Q \setminus \{\underline{s}\}} \subset I_{\underline{s}} \cap I_{\tilde{Q} \setminus \{\underline{s}\}} = I_{\underline{s}} \cap I_{k-1} = \sum_{\underline{t} \subset \underline{s}, \underline{t} \in \mathcal{S}_X(k-1)} I_{\underline{t}}.$$

On the other hand it is clear that

$$I_{\underline{s}} \cap I_{Q \setminus \{\underline{s}\}} \supset \sum_{\underline{t} \subset \underline{s}, \underline{t} \in \mathcal{S}_X(k-1)} I_{\underline{t}}$$

and our claim follows. \square

3. EQUIVARIANT COHOMOLOGY

3.1. Equivariant cohomology of $M_{X, \geq k}$. Let G be a compact torus. Given a G space M , we denote for simplicity by $H_G^*(M)$ the G equivariant cohomology $H_G^*(M, \mathbb{R})$ of M with real coefficients.

For a character $a \in \Lambda$, we denote by L_a the one dimensional complex G module on which G acts via a . Given a list X in Λ , we set

$$M_X = \oplus_{a \in X} L_a.$$

Our purpose is to compute the equivariant cohomology of various G stable open sets in M_X .

To begin with, since M_X is a vector space, $H_G^*(M_X)$ equals the equivariant cohomology of a point and thus $H_G^*(M_X) = S[\mathfrak{g}^*]$, and $\mathfrak{g}^* = H_G^2(M_X)$.

Let X and M_X be as before and Y a sublist of X . We have $M_Y \subset M_X$.

Lemma 3.2. $H_G^*(M_X \setminus M_Y) = S[\mathfrak{g}^*]/(d_{X \setminus Y})$.

Proof. Since $M_X \setminus M_Y = (M_{X \setminus Y} \setminus 0) \times M_Y$, we have $H_G^*(M_X \setminus M_Y) \cong H_G^*(M_{X \setminus Y} \setminus 0)$. Moreover, the long exact sequence of the pair $(M_{X \setminus Y}, 0)$ and the definition of the equivariant Euler class yield $H_G^*(M_{X \setminus Y} \setminus 0) \cong H_G^*(M_{X \setminus Y})/(d_{X \setminus Y})$. \square

Take a subset $Q \subset \mathcal{S}_X$ of rational subspaces and set

$$\mathcal{A}_Q = M_X \setminus \cup_{\underline{s} \in Q} M_{X \cap \underline{s}}.$$

Theorem 3.3. $H_G^*(\mathcal{A}_Q)$ is isomorphic as a graded ring to $S[\mathfrak{g}^*]/I_Q$.

In particular \mathcal{A}_Q has no G equivariant odd cohomology.

Proof. Let us add to Q all the rational subspaces \underline{t} which are contained in at least one of the elements of Q . In this way, we get a new subset $\overline{Q} \supset Q$ which is now admissible and is such that $\mathcal{A}_Q = \mathcal{A}_{\overline{Q}}$. Also it is clear that $I_Q = I_{\overline{Q}}$.

Having made this remark, we may without loss of generality assume that Q is admissible. If $Q = \emptyset$, then $\mathcal{A}_Q = M_X$, the ideal $I_Q = \{0\}$ and there is nothing to prove. Thus we can proceed by induction on the cardinality of Q and assume that Q is nonempty.

Take $\{\underline{s}\}$ maximal in Q . Notice that $Q \setminus \{\underline{s}\}$ is also admissible. Furthermore the set

$$\mathcal{S}_{<\underline{s}} = \{\underline{t} \in \mathcal{S}_X \mid \underline{t} \subsetneq \underline{s}\}$$

is also admissible and strictly contained in Q .

We have

$$\mathcal{A}_Q = \mathcal{A}_{Q \setminus \{\underline{s}\}} \cap (M_X \setminus M_{X \cap \underline{s}})$$

and

$$\mathcal{A}_{Q \setminus \{\underline{s}\}} \cup (M_X \setminus M_{X \cap \underline{s}}) = \mathcal{A}_{\mathcal{S}_{<\underline{s}}}.$$

Thus, by induction, we have

$$(8) \quad H_G^*(\mathcal{A}_{Q \setminus \{\underline{s}\}}) = S[\mathfrak{g}^*]/I_{Q \setminus \{\underline{s}\}}, \quad H_G^*(\mathcal{A}_{Q \setminus \{\underline{s}\}} \cup (M_X \setminus M_{X \cap \underline{s}})) = S[\mathfrak{g}^*]/I_{\mathcal{S}_{<\underline{s}}}.$$

Set $Y := X \cap \underline{s}$. Consider the homomorphism

$$\psi : H_G^*(\mathcal{A}_{Q \setminus \{\underline{s}\}} \cup (M_X \setminus M_{X \cap \underline{s}})) \rightarrow H_G^*(\mathcal{A}_{Q \setminus \{\underline{s}\}}) \oplus H_G^*(M_X \setminus M_Y)$$

induced by inclusion. Using the isomorphisms (8) and Lemma 3.2, we get a commutative diagram

$$\begin{array}{ccc} H_G^*(\mathcal{A}_{Q \setminus \{\underline{s}\}} \cup (M_X \setminus M_{X \cap \underline{s}})) & \xrightarrow{\psi} & H_G^*(\mathcal{A}_{Q \setminus \{\underline{s}\}}) \oplus H_G^*(M_X \setminus M_Y) \\ \simeq \downarrow & & \simeq \downarrow \\ S[\mathfrak{g}^*]/I_{\mathcal{S}_{<\underline{s}}} & \longrightarrow & S[\mathfrak{g}^*]/I_{Q \setminus \{\underline{s}\}} \oplus S[\mathfrak{g}^*]/(d_{X \setminus Y}) \end{array}$$

where the vertical arrows are isomorphisms. Now by Proposition 2.8 2)

$$I_{\underline{s}} \cap I_{Q \setminus \{\underline{s}\}} = I_{\underline{s}} \cap I_{k-1} = \sum_{\underline{t} \subset \underline{s}, \underline{t} \in \mathcal{S}_X(k-1)} I_{\underline{t}} = I_{\mathcal{S}_{<\underline{s}}}.$$

Thus ψ is injective. We immediately deduce from the Mayer-Vietoris sequence that the homomorphism

$$\phi : H_G^*(\mathcal{A}_{Q \setminus \{\underline{s}\}}) \oplus H_G^*(M_X \setminus M_Y) \rightarrow H_G^*(\mathcal{A}_Q)$$

is surjective and that $H_G^*(\mathcal{A}_Q) \simeq S[\mathfrak{g}^*]/I_Q$ as desired. \square

Remark 3.4. There is a parallel theorem for the algebraic counterpart of equivariant cohomology, that is the equivariant Chow ring (see Edidin and Graham [10]).

3.5. Equivariant cohomology of $M_{X, \geq k}$. Let $s := \dim(G)$ and let us look at some special cases of Theorem 3.3. We always assume that X spans the s -dimensional space \mathfrak{g}^* , which is equivalent to assume that the generic point of M_X has a finite stabilizer.

If $Q = \mathcal{S}_X(k-1)$,

$$\mathcal{A}_{\mathcal{S}_X(k-1)} = M_X \setminus \bigcup_{\mathfrak{s} \in \mathcal{S}_X(k-1)} M_{X \cap \mathfrak{s}} := M_{X, \geq k}$$

is the set of points whose orbits have dimension at least k .

Definition 3.6. For $k = s$, $M_{X, \geq s}$ is the open set of points with finite stabilizer that we also denote by M_X^{fin} .

Corollary 3.7. *The equivariant cohomology of $M_{X, \geq k}$ is isomorphic as a graded algebra to $S[\mathfrak{g}^*]$ modulo the ideal I_{k-1} . In particular $H_G^*(M_X^{fin}) = S[\mathfrak{g}^*]/I_X$ with I_X the ideal generated by the elements d_Y as Y runs over the cocircuits.*

Remark 3.8. Assume that X spans $\Lambda \subset \mathfrak{g}^*$ and that the cone $C(X) \subset \mathfrak{g}^*$ of linear combinations of the vectors in X with non negative coefficients is acute. Consider the complexified torus $G_{\mathbb{C}}$. The list X gives an embedding $G \rightarrow (S_1)^m$ and its complexification $G_{\mathbb{C}} \rightarrow (\mathbb{C}^*)^m$, so we may consider the torus $T = (S_1)^m/G$ and its complexification $T_{\mathbb{C}} = (\mathbb{C}^*)^m/G_{\mathbb{C}}$.

Write $z \in M_X$ as $z = \sum_a z_a$ with $z_a \in L_a$. Choose $\xi \in C(X)$ not lying in any rational hyperplane (in this case we say that ξ is regular). Then the set $P_{\xi} := \{z \in M_X \mid \sum_a |z_a|^2 a = \xi\}$ is smooth, contained in M_X^{fin} , and P_{ξ}/G is a toric variety for the complex torus $T_{\mathbb{C}}$. As a $T_{\mathbb{C}}$ -variety, P_{ξ}/G depends only on the connected component of the set of regular points containing ξ . Furthermore P_{ξ}/G is projective and rationally smooth.

Generators and relations for the ring $H_G^*(P_{\xi}) = H^*(P_{\xi}/G)$ are well known (see for example [4]). In particular, $S(\mathfrak{g}^*)$ surjects on $H^*(P_{\xi}/G)$. Consider the restriction map $H_G^*(M_X^{fin}) \rightarrow H_G^*(P_{\xi})$. Thus this map is surjective for any regular ξ and its kernel (which depends upon the connected component of the set of regular points containing ξ) is generated by the polynomials $d_{X \setminus \sigma} \in S(\mathfrak{g}^*)$, where $\sigma \subset X$ runs over the bases of \mathfrak{g}^* such that ξ is not in the cone generated by σ .

Remark 3.9. It may be interesting to observe that to X , as to any matroid, is associated a two variable polynomial, the Tutte polynomial, that describes the statistics of external and internal activity. Then the statistic of external activity gives rise to the Betti numbers of equivariant cohomology of M_X^{fin} while from internal activity one deduces the characteristic polynomial that describes Betti numbers of the complement of the complex hyperplane arrangement deduced from X . A direct topological interpretation of the Tutte polynomial has been recently obtained in [11].

4. EQUIVARIANT COHOMOLOGY OF T_G^*M

4.1. **The space $D(X)$.** In order to perform our cohomology computations, we need first to introduce some new spaces. We keep the notation of the previous sections.

Given $a \in \mathfrak{g}^*$, let us denote by ∂_a the derivative in the a direction. We identify $S[\mathfrak{g}^*]$ to the space of differential operators with constant coefficients on \mathfrak{g}^* .

To a cocircuit Y , we associate the differential operator $\partial_Y := \prod_{a \in Y} \partial_a$.

Definition 4.2. The space $D(X)$ is given by

$$(9) \quad D(X) := \{f \in S[\mathfrak{g}] \mid \partial_Y f = 0, \text{ for every cocircuit } Y\}.$$

The space $D(X)$ is stable by the action of $S[\mathfrak{g}^*]$.

Notice that, by its definition:

Remark 4.3. $D(X)$ is the (graded) vector space dual to the algebra $D^*(X) = S[\mathfrak{g}^*]/I_X$, that is the cohomology ring $H_G^*(M_X^{fin})$ by Corollary 3.7.

To be consistent with grading in cohomology, we double the degrees in $S[\mathfrak{g}]$ and hence in $D(X)$ and we set for each $i \geq 0$, $D(X)^{2i+1} = \{0\}$.

Using the Lebesgue measure associated to the lattice Λ , we will in what follows freely identify polynomial functions on \mathfrak{g}^* with polynomial densities on \mathfrak{g}^* .

The polynomials in $D(X)$, dual to the algebra $D^*(X) := S[\mathfrak{g}^*]/I_X$, can be naturally interpreted as Laplace–Fourier transforms of the finite dimensional space $\hat{D}(X)$ of those generalized functions which vanish on the functions vanishing on the scheme V_X .

Denote by $\mathcal{S}'(\mathfrak{g}^*)$ the space of tempered distributions on \mathfrak{g}^* . Assume now that there is an element $x \in \mathfrak{g}$ such that $\langle x, a \rangle > 0$ for every a in X . Recall that the *multivariate spline* T_X is the tempered distribution defined by:

$$(10) \quad \langle T_X \mid f \rangle = \int_0^\infty \dots \int_0^\infty f\left(\sum_{i=1}^m t_i a_i\right) dt_1 \dots dt_m.$$

Its Laplace transform is $d_X^{-1} := 1/\prod_{a \in X} a$. Notice that, if $a \in X$,

$$(11) \quad T_X = T_a * T_{X \setminus a}, \quad \partial_a T_X = T_{X \setminus a}, \implies \partial_X T_X = T_\emptyset = \delta_0.$$

Let \underline{r} be a vector subspace in \mathfrak{g}^* . We have an embedding $j : \mathcal{S}'(\underline{r}) \rightarrow \mathcal{S}'(\mathfrak{g}^*)$ by $j(\phi)(f) = \phi(f|_{\underline{r}})$ for any $\phi \in \mathcal{S}'(\underline{r})$, f a Schwartz function on \mathfrak{g}^* . We denote the image $j(\mathcal{S}'(\underline{r}))$ by $\mathcal{S}'(\mathfrak{g}^*, \underline{r})$ (sometimes we even identify $\mathcal{S}'(\underline{r})$ with $\mathcal{S}'(\mathfrak{g}^*, \underline{r})$ if there is no ambiguity). We next define the vector space:

Definition 4.4.

$$(12) \quad \mathcal{G}(X) := \{f \in \mathcal{S}'(\mathfrak{g}^*) \mid \partial_{X \setminus \underline{r}} f \in \mathcal{S}'(\mathfrak{g}^*, \underline{r}), \text{ for all } \underline{r} \in \mathcal{S}_X\}.$$

Example 4.5. Let $G = S^1$ and identify Λ with \mathbb{Z} and \mathfrak{g}^* with \mathbb{R} . Let $X = 1^{k+1} = \underbrace{(1, 1, \dots, 1)}_{k+1}$.

Then there are two rational subspaces: \mathbb{R} and the origin. The only cocircuit is X itself and $\partial_X = \frac{d^{k+1}}{dx^{k+1}}$. The space $D(X)$ consists of the polynomials of degree $\leq k$ and $T_X = x^k/k!$ if $x \geq 0$ and 0 otherwise. It is easy to see that $\mathcal{G}(X) = D(X) \oplus \mathbb{R}T_X$.

We are now going to recall a few properties of $\mathcal{G}(X)$ (see also [7]). For this, given a list of non zero vectors Z in \mathfrak{g}^* , we consider the dual hyperplane arrangement, $a^\perp \subset \mathfrak{g}$, $a \in Z$. Any connected component F of the complement of this arrangement is called a *regular face* for Z . An element $\phi \in F$ decomposes $Z = A \cup B$ where ϕ is positive on A and negative on B . This decomposition depends only upon F . We define

$$(13) \quad T_Z^F = (-1)^{|B|} T_{(A, -B)}.$$

Notice that T_Z^F is supported on the cone $C(A, -B)$ of non negative linear combinations of the vectors in the list $(A, -B)$.

Take the subset $\mathcal{S}_X(i)$ of subspaces $\underline{r} \in \mathcal{S}_X$ of dimension i . Consider $\partial_{X \setminus \underline{r}}$ as an operator on $\mathcal{G}(X)$ with values in $\mathcal{S}'(\mathfrak{g}^*, \underline{r})$. Define the spaces

$$(14) \quad \mathcal{G}(X)_i := \cap_{\underline{t} \in \mathcal{S}_X(i-1)} \ker(\partial_{X \setminus \underline{t}}).$$

Notice that by definition $\mathcal{G}(X)_0 = \mathcal{G}(X)$, that $\mathcal{G}(X)_{\dim \mathfrak{g}^*}$ is the space $D(X)$ and that $\mathcal{G}(X)_{i+1} \subseteq \mathcal{G}(X)_i$.

Remark 4.6. Consider a polynomial density $g \in D(X \cap \underline{r})$, a face $F_{\underline{r}}$ defining $X \setminus \underline{r} = A \cup B$ and $T_{X \setminus \underline{r}}^{F_{\underline{r}}}$. The convolution $T_{X \setminus \underline{r}}^{F_{\underline{r}}} * g$ is well defined since, for any $z \in \mathfrak{g}^*$, the set of pairs $x \in C(A, -B), y \in \underline{r}$ with $x + y = z$ is compact.

Lemma 4.7. *Let $\underline{r} \in \mathcal{S}_X(i)$.*

- i) The image of $\partial_{X \setminus \underline{r}}$ restricted to $\mathcal{G}(X)_i$ is contained in $D(X \cap \underline{r})$.*
- ii) Take rational subspaces \underline{t} and \underline{r} . For any $g \in D(X \cap \underline{r})$,*

$$(15) \quad \partial_{X \setminus \underline{t}}(T_{X \setminus \underline{r}}^{F_{\underline{r}}} * g) = (\partial_{(X \setminus \underline{t}) \setminus \underline{r}} T_{X \setminus \underline{r}}^{F_{\underline{r}}}) * (\partial_{(X \cap \underline{r}) \setminus (\underline{t} \cap \underline{r})} g).$$

- iii) If g is in $D(X \cap \underline{r})$, then $T_{X \setminus \underline{r}}^{F_{\underline{r}}} * g \in \mathcal{G}(X)_i$.*

Proof. *i)* First we know, by the definition of $\mathcal{G}(X)$, that $\partial_{X \setminus \underline{r}} \mathcal{G}(X)_i$ is contained in the space $\mathcal{S}'(\mathfrak{g}^*, \underline{r})$. Let \underline{t} be a rational hyperplane of \underline{r} , so that \underline{t} is of dimension $i - 1$. By definition, we have that for every $f \in \mathcal{G}(X)_i$

$$0 = \prod_{a \in X \setminus \underline{t}} \partial_a f = \prod_{a \in (X \cap \underline{r}) \setminus \underline{t}} \partial_a \partial_{X \setminus \underline{r}} f.$$

This means that $\partial_{X \setminus \underline{r}} f$ satisfies the differential equations given by the cocircuits of $X \cap \underline{r}$, that is, it lies in $D(X \cap \underline{r})$.

ii) We have that $\partial_{X \setminus \underline{t}} = \partial_{(X \setminus \underline{t}) \cap \underline{r}} \partial_{(X \setminus \underline{t}) \setminus \underline{r}}$ but $\partial_{(X \setminus \underline{t}) \cap \underline{r}} = \partial_{(X \cap \underline{r}) \setminus (\underline{t} \cap \underline{r})}$. Thus

$$\partial_{X \setminus \underline{t}}(T_{X \setminus \underline{r}}^{F_{\underline{r}}} * g) = (\partial_{(X \setminus \underline{t}) \setminus \underline{r}} T_{X \setminus \underline{r}}^{F_{\underline{r}}}) * (\partial_{(X \cap \underline{r}) \setminus (\underline{t} \cap \underline{r})} g)$$

as desired.

iii) If \underline{t} does not contain \underline{r} , we get that $(\partial_{(X \cap \underline{r}) \setminus (\underline{t} \cap \underline{r})} g) = 0$ and hence, by (15),

$$\partial_{X \setminus \underline{t}}(T_{X \setminus \underline{r}}^{F_{\underline{r}}} * g) = 0.$$

□

Consider the map $\mu_i : \mathcal{G}(X)_i \rightarrow \oplus_{\underline{r} \in \mathcal{S}_X(i)} D(X \cap \underline{r})$ given by

$$\mu_i f := \oplus_{\underline{r} \in \mathcal{S}_X(i)} \partial_{X \setminus \underline{r}} f$$

and the map $\mathbf{P}_i : \oplus_{\underline{r} \in \mathcal{S}_X(i)} D(X \cap \underline{r}) \rightarrow \mathcal{G}(X)_i$ given by

$$\mathbf{P}_i(\oplus g_{\underline{r}}) := \sum T_{X \setminus \underline{r}}^{F_{\underline{r}}} * g_{\underline{r}}.$$

Theorem 4.8. *The sequence*

$$0 \longrightarrow \mathcal{G}(X)_{i+1} \longrightarrow \mathcal{G}(X)_i \xrightarrow{\mu_i} \oplus_{\underline{r} \in \mathcal{S}_X(i)} D(X \cap \underline{r}) \longrightarrow 0$$

is exact. Furthermore, the map \mathbf{P}_i provides a splitting of this exact sequence, i.e. $\mu_i \mathbf{P}_i = \text{Id}$.

Proof. By definition, $\mathcal{G}(X)_{i+1}$ is the kernel of μ_i , thus we only need to show that $\mu_i \mathbf{P}_i = \text{Id}$. Given $\underline{r} \in \mathcal{S}_X(i)$ and $g \in D(X \cap \underline{r})$, by Formula (15) we have $\partial_{X \setminus \underline{r}}(T_{X \setminus \underline{r}}^{F_{\underline{r}}} * g) = g$. If instead we take another subspace $\underline{t} \neq \underline{r}$ of $\mathcal{S}_X(i)$, then $\underline{r} \cap \underline{t}$ is a proper subspace of \underline{t} . As we have seen above, if $g \in D(X \cap \underline{r})$, $\partial_{X \setminus \underline{t}}(T_{X \setminus \underline{r}}^{F_{\underline{r}}} * g) = 0$. Thus, given a family $g_{\underline{r}} \in D(X \cap \underline{r})$, the function $f = \sum_{\underline{t} \in \mathcal{S}_X(i)} T_{X \setminus \underline{t}}^{F_{\underline{t}}} * g_{\underline{t}}$ is such that $\partial_{X \setminus \underline{r}} f = g_{\underline{r}}$ for all $\underline{r} \in \mathcal{S}_X(i)$. This proves our claim that $\mu_i \mathbf{P}_i = \text{Id}$. □

Putting together these facts, we immediately get

Theorem 4.9. *Choose, for every rational space \underline{r} , a regular face $F_{\underline{r}}$ for $X \setminus \underline{r}$. Then:*

$$(16) \quad \mathcal{G}(X) = \oplus_{\underline{r} \in \mathcal{S}_X} T_{X \setminus \underline{r}}^{F_{\underline{r}}} * D(X \cap \underline{r}).$$

Corollary 4.10. *The dimension of $\mathcal{G}(X)$ equals the number of sublists of X which are linearly independent.*

Proof. . This follows immediately from (16) and the fact (see for example [5] Theorem 11.8) that $D(X)$ has dimension equal to the number of bases which can be extracted from X . □

We define

$$(17) \quad \tilde{\mathcal{G}}(X) = S[\mathfrak{g}^*] \mathcal{G}(X), \quad \tilde{\mathcal{G}}_i(X) = S[\mathfrak{g}^*] \mathcal{G}_i(X)$$

where the elements in $S[\mathfrak{g}^*]$ act on distributions as differential operators with constant coefficients.

Remark 4.11. If we set

$$D^{\mathfrak{g}}(X \cap \underline{r}) = S[\mathfrak{g}^*]D(X \cap \underline{r}) \cong S[\mathfrak{g}^*] \otimes_{S[(\mathfrak{g}/\mathfrak{g}_{\underline{r}})^*]} D(X \cap \underline{r}),$$

Theorem 4.8, together with the fact that the maps μ_i and P_i extend to $S[\mathfrak{g}^*]$ -module maps (which we denote by the same letter), immediately implies that we have an exact sequence of $S[\mathfrak{g}^*]$ -modules

$$0 \rightarrow \tilde{\mathcal{G}}_{i+1}(X) \rightarrow \tilde{\mathcal{G}}_i(X) \xrightarrow{\mu_i} \bigoplus_{\underline{r} \in \mathcal{S}_X(i)} D^{\mathfrak{g}}(X \cap \underline{r}) \rightarrow 0.$$

Furthermore one can give generators for $\tilde{\mathcal{G}}(X)$ as a $S[\mathfrak{g}^*]$ -module as follows:

Theorem 4.12.

$$\tilde{\mathcal{G}}(X) = \sum_F S[\mathfrak{g}^*]T_X^F$$

as F runs over all regular faces for X .

Proof. Denote by M the $S[\mathfrak{g}^*]$ module generated by the elements T_X^F , as F runs on all regular faces. In general, from the description of $\tilde{\mathcal{G}}(X)$ given in Formula (16), it is enough to prove that elements of the type $T_{X \setminus \underline{r}}^{F_{\underline{r}}} * g$ with $g \in D(X \cap \underline{r})$ are in M . As $D(X \cap \underline{r}) \subset \mathcal{G}(X \cap \underline{r})$, it is sufficient to prove by induction that each element $T_{X \setminus \underline{r}}^{F_{\underline{r}}} * T_{X \cap \underline{r}}^K$ is in M , where K is any regular face for the system $X \cap \underline{r}$. We choose a linear function u_0 in the face $F_{\underline{r}}$. Thus u_0 vanishes on \underline{r} and is non zero on every element $a \in X$ not in \underline{r} . We choose a linear function u_1 such that the restriction of u_1 to \underline{r} lies in the face K . In particular, u_1 is non zero on every element $a \in X \cap \underline{r}$. We can choose ϵ sufficiently small such that $u := u_0 + \epsilon u_1$ is non zero on every element $a \in X$. Then $u_0 + \epsilon u_1$ defines a regular face F . We see that a vector $a \in X \setminus \underline{r}$ is positive for u if and only if it is positive for u_0 , similarly a vector $a \in X \cap \underline{r}$ is positive for u if and only if it is positive for u_1 , hence from (11), and the definition (13), it follows that $T_{X \setminus \underline{r}}^{F_{\underline{r}}} * T_{X \cap \underline{r}}^K$ is equal to T_X^F . \square

This construction has a discrete counterpart, thoroughly studied in [7] and related to the study of the index of transversally elliptic operators and of computations in equivariant K-theory in which differential operators are replaced by difference operators.

5. EQUIVARIANT COHOMOLOGY WITH COMPACT SUPPORTS OF T^*M_X .

5.1. Equivariant cohomology with compact supports and the infinitesimal index. Let G be a compact Lie group, in [8] we have introduced a de Rham model for the equivariant cohomology $H_{G,c}^*(Z)$ with compact supports of a G -stable closed subset $Z \subset N$ of a G -manifold N . A representative of an element in $H_{G,c}^*(Z)$ is a compactly supported equivariant form on N such that $D\alpha$ is zero on a neighborhood of Z . Two representatives

α_1, α_2 agree if $\alpha_1 - \alpha_2 = D\beta + \gamma$ where β, γ are compactly supported and γ vanishes on a neighborhood of Z .

Furthermore assume that we have a G -equivariant one form σ on N called an action form. We define the corresponding moment map $\mu : N \rightarrow \mathfrak{g}^*$ by setting for any $u \in \mathfrak{g}$, $n \in N$, $\mu(n)(x) := -\langle \sigma, v_x \rangle(n)$, v_x being the vector field on N corresponding to x . Set $\Omega(x) := D\sigma(x) = d\sigma + \mu(x)$ (D is the equivariant differential).

If we take as Z the zeroes $N^0 = \mu^{-1}(0)$ of the moment map, we have then defined a map of $S[\mathfrak{g}^*]$ -modules

$$\text{index} : H_{G,c}^*(N^0) \rightarrow \mathcal{S}'(\mathfrak{g}^*)^G$$

called infinitesimal index. For $\alpha(x)$ a form giving a cohomology class $[\alpha] \in H_{G,c}^*(N^0)$ and f a test function we have:

$$(18) \quad \langle \text{index}_G^\mu([\alpha]), f \rangle = \lim_{s \rightarrow \infty} \int_N \int_{\mathfrak{g}} e^{is\Omega(x)} \alpha(x) \hat{f}(x) dx$$

This is a well defined map from $H_{G,c}^*(N^0)$ to invariant distributions on \mathfrak{g}^* . We refer to [8] for the proof of most of the properties of $H_{G,c}^*(Z)$ and of the infinitesimal index which we are going to use in what follows.

We are going to study the case in which we start with a G -manifold M . We set $N = T^*M$ and we take the canonical one form σ . In this case it follows immediately from the definitions that $(T^*M)^0$ equals the space T_G^*M whose fiber over a point $x \in M$ is formed by all the cotangent vectors $\xi \in T_x^*M$ which vanish on the tangent space to the orbit of x under G , in the point x . Thus each fiber $(T_G^*M)_x$ is a linear subspace of T_x^*M . In general the dimension of $(T_G^*M)_x$ is not constant and this space is not a vector bundle on M .

5.2. Connection forms and the Chern-Weil map. We shall use a fundamental notion in Cartan's theory of equivariant cohomology. Let us recall

Definition 5.3. Given an action of a compact Lie group G on a manifold P with finite stabilizers, a *connection form* is a G -invariant one form $\omega \in \mathcal{A}^1(P) \otimes \mathfrak{g}$ with coefficients in the Lie algebra of G such that $-\iota_x \omega = x$ for all $x \in \mathfrak{g}$.

If on P with free G action we also have a commuting action of another group L , it is easy to see that there exists a $L \times G$ invariant connection form $\omega \in \mathcal{A}^1(P) \otimes \mathfrak{g}$ on P for the free action of G .

Let $Q := P/G$. Define the curvature R and the L -equivariant curvature R_y of the bundle $P \rightarrow Q$ by

$$(19) \quad R := d\omega + \frac{1}{2}[\omega, \omega], \quad R_y := -i_y \omega + R.$$

We have the Chern-Weil map $c : S[\mathfrak{g}^*] \rightarrow H_G^*(P) \cong H^*(P/G)$ defined by $p \mapsto [p(R)]$ (see [8] p.8). Through this map we give to $H^*(P/G)$ and $H_c^*(P/G)$ a $S[\mathfrak{g}^*]$ module structure.

Proposition 5.4. *If G acts freely (or with finite stabilizers) on a manifold P , the Poincaré duality for $Q = P/G$ commutes with the $S[\mathfrak{g}^*]^G$ -module structures.*

Proof. This depends upon the fact that the $S[\mathfrak{g}^*]^G$ -module structure of $H^*(Q)$ comes from the Chern–Weil morphism $S[\mathfrak{g}^*]^G \rightarrow H^{even}(Q)$ determined by the bundle. The $S[\mathfrak{g}^*]^G$ -module structure of $H_c^*(Q)$ comes from the same morphism and the fact that $H_c^*(Q)$ is a $H^*(Q)$ module under multiplication and finally that duality is given by integration formula $\int_Q \alpha \wedge \beta$ with α, β closed and β with compact support. \square

5.5. The equivariant cohomology of $T_G^* M_X^{fin}$. Our task is now to use the infinitesimal index to compute the equivariant cohomology with compact supports of $T_G^* M_X$ and more generally of $T_G^* M_{X, \geq k}$. Notice that if we consider ordinary equivariant cohomology, it is immediate by G -homotopy equivalence to deduce

Proposition 5.6. *The equivariant cohomology of the space $T_G^* M_{X, \geq k}$ equals that of $M_{X, \geq k}$ for all k .*

We have already remarked that, in the case $k = s$, we have $M_{X, \geq k} = M_X^{fin}$ and that $H_G^*(M_X^{fin}) = D^*(X)$. Now, since G acts on M_X^{fin} with finite stabilizers, and we use cohomology with real coefficients, we get that $H_G^*(M_X^{fin}) = H^*(M_X^{fin}/G)$ and by Poincaré duality

$$(20) \quad H_{G,c}^h(M_X^{fin}) = H_c^h(M_X^{fin}/G) = (H^{2|X|-s-h}(M_X^{fin}/G))^* = D^{2|X|-s-h}(X).$$

Now, in order to compute the equivariant cohomology with compact supports of $T_G^* M_X^{fin}$, we need some well known general considerations.

Let N be a G -manifold, \mathcal{M} be a G -equivariant vector bundle on N of rank r with projection $p : \mathcal{M} \rightarrow N$. Then (see [12]), there is an equivariant Thom form $\tau_{\mathcal{M}}$, which can be taken supported in any arbitrarily small disk bundle around N in \mathcal{M} , such that in particular:

Proposition 5.7. *The map*

$$C : H_{G,c}^*(N) \rightarrow H_{G,c}^{*+r}(\mathcal{M})$$

defined by $C(\alpha) = p^(\alpha) \wedge \tau_{\mathcal{M}}$ is an isomorphism*

Let Z be an oriented G manifold and $s : N \hookrightarrow Z$ a G -stable oriented submanifold. Assume that N has an action form with moment map μ and that Z is equipped with an action form such that the associated moment map μ_Z extends μ . Thus $Z^0 \cap N = N^0$.

We have used the Thom form in [8] in order to define a map

$$s! : H_{G,c}^*(N^0) \rightarrow H_{G,c}^*(Z^0).$$

In particular we can apply this when $Z = N \times M_B$, $s : N \rightarrow N \times M_B$ is the embedding of the 0 section and M_B is the linear representation associated

with some list B of non zero vectors in Λ equipped with an action form with the origin lying in M_B^0 . Then we take on $N \times M_B$ the action form given by the sum of the action forms on N and M_B and we get the map

$$s! : H_{G,c}^*(N^0) \rightarrow H_{G,c}^*((N \times M_B)^0).$$

We have

Proposition 5.8. *If $[\lambda] \in H_{G,c}^*((N \times M_B)^0)$, then:*

$$s!s^*([\lambda]) = d_B[\lambda].$$

Proof. We first want a Poincaré Lemma for a G manifold N and a vector space M_B with G action.

Consider the map $q : N \times M_B \times [0, 1] \rightarrow N \times M_B$, $q(x, y, t) = (x, ty)$ and the maps $i_t : N \times M_B \rightarrow N \times M_B \times [0, 1]$, $i_t(x, y) = (x, y, t)$.

Given an equivariant form $\lambda = \lambda(x, y)$, $x \in N, y \in M_B$ on $N \times M_B$, define the forms $\tilde{\lambda}, \bar{\lambda}$ on $N \times M_B \times [0, 1]$ by

$$q^*\lambda = \tilde{\lambda} + dt \wedge \bar{\lambda}$$

where $\tilde{\lambda}$ does not contain dt and can be thought of as the form $q_t^*\lambda$, and now $q_t : (x, y) \rightarrow (x, ty)$. Set

$$I(\lambda) := \int_0^1 i_t^* \bar{\lambda} dt.$$

We claim that we have the homotopy formula

$$(21) \quad \lambda(x, y) - \lambda(x, 0) = DI\lambda + ID\lambda.$$

In fact

$$DI(\lambda) := \int_0^1 i_t^* D\bar{\lambda} dt, \quad q^*D\lambda = D\tilde{\lambda} - dt \wedge D\bar{\lambda}.$$

Write $D\tilde{\lambda} = \omega + dt \wedge \eta$, so that $ID\lambda = \int_0^1 i_t^*(\eta - D\bar{\lambda})dt$ gives

$$DI(\lambda) = -ID\lambda + \int_0^1 i_t^* \eta dt.$$

If we think of $\tilde{\lambda}$ as a form on $N \times M_B$ depending on t , we see that $\eta = \frac{d}{dt}\tilde{\lambda} = \frac{d}{dt}q_t^*\lambda$. It follows that

$$\int_0^1 i_t^* \eta dt = q_1^*\lambda - q_0^*\lambda.$$

We now multiply Equation (21) above by a Thom form τ for the trivial bundle $p : N \times M_B \rightarrow N$ which is the pull back of a Thom form τ_0 on M_B under the projection $N \times M_B \rightarrow M_B$. We may assume the support of τ as close to N as we wish, that is in $N \times B_\epsilon$ with B_ϵ a ball of radius ϵ centered in the origin. In particular, take a form λ such that $D\lambda$ has support K disjoint from $(N \times M_B)^0$. For such a form, the support of $ID\lambda$ is contained in the set of points (x, y) such that the segment (x, ty) , $t \in [0, 1]$, intersects K . The support of $\tau ID\lambda$ is contained in the previous set of points (x, y) but with the further requirement that $y \in B_\epsilon$.

Then, if ϵ is small, we can make sure that the set $\cup_{t \in [0,1]} t(K \cap (N \times B_\epsilon))$ is still disjoint from $(N \times M_B)^0$.

We deduce that

$$\tau\lambda(x, y) - \tau\lambda(x, 0) = D\tau I\lambda + \tau ID\lambda,$$

the forms in this equality have all compact support and moreover the support of $\tau ID\lambda$ is disjoint from $(N \times M_B)^0$.

This implies that $[\tau\lambda(x, y)] = [\tau\lambda(x, 0)]$ in the cohomology $H_{G,c}^*((N \times M_B)^0)$.

Now by definition $\lambda(x, 0) = p^*s^*\lambda$, so $[\tau\lambda(x, 0)] = s_!s^*[\lambda]$.

The inclusion of the origin gives an algebra isomorphism between $H_G^*(M_B)$ and $H_G^*(pt) = S[\mathfrak{g}^*]$ and the image of the class of τ_0 is the Euler class of M_B which is d_B . Thus $[\tau\lambda] = [d_B\lambda]$. \square

Remark 5.9. Notice that, if we assume that the moment map on M_B equals 0, then $(N \times M_B)^0 = N^0 \times M_B$ and the map $s_!$ is just the Thom isomorphism between $H_{G,c}^*(N^0)$ and $H_{G,c}^*(N^0 \times M_B)$.

Let us now go back to our computations. The projection $p : T_G^*M_X^{fin} \rightarrow M_X^{fin}$ is a real vector bundle of rank $2|X| - s$ so that, applying Proposition 5.7, we get $H_{G,c}^*(T_G^*M_X^{fin}) = H_{G,c}^{*+2|X|-s}(M_X^{fin})$. Thus putting together this with (20), we get

Proposition 5.10. *As a graded $S[\mathfrak{g}^*]$ -module,*

$$(22) \quad H_{G,c}^*(T_G^*M_X^{fin}) \simeq D^{4|X|-2s-*}(X).$$

*In particular $T_G^*M_X^{fin}$ has no equivariant odd cohomology with compact supports.*

Proof. We apply Proposition 5.4 and deduce that $H_{G,c}^*(T_G^*M_X^{fin})$ is isomorphic to the dual of $H_G^*(T_G^*M_X^{fin})$ as $S[\mathfrak{g}^*]$ -modules, where the dual structure is given by $\langle a\phi, u \rangle = \langle \phi, au \rangle$, $a \in S[\mathfrak{g}^*]$, $\phi \in H_G^*(T_G^*M_X^{fin})^*$, $u \in H_G^*(T_G^*M_X^{fin})$. The statement now follows from Remark 4.3. \square

5.11. The equivariant cohomology of $T_G^*M_{X,\geq i}$. It is now interesting to interpret formula (22) via the theory of the infinitesimal index. To do this, we need to recall a few facts. As we have already remarked, the action of G on $T_G^*M_X^{fin}$ is essentially free, so, denoting by Q the quotient $T_G^*M_X^{fin}/G$, we can take an equivariant \mathfrak{g} -valued curvature form R for the map $T_G^*M_X^{fin} \rightarrow Q$.

We have the Chern-Weil map $c : S[\mathfrak{g}^*] \rightarrow H_G^*(T_G^*M_X^{fin})$ defined by $p \mapsto [p(R)]$.

Take a closed equivariant form with compact support γ on Q . We can apply the Theory of the infinitesimal index to the class $[\gamma] \in H_c^*(Q) \simeq$

$H_{G,c}^*(T_G^*M_X^{fin})$. By Proposition 4.20 of [8] the infinitesimal index is given by the polynomial density on \mathfrak{g}^*

$$(23) \quad \left(\int_Q \gamma e^{i\langle R, \xi \rangle} \right) d\xi.$$

We have

Theorem 5.12. *The map infdex is a graded isomorphism as $S[\mathfrak{g}^*]$ -modules of $H_{G,c}^*(T_G^*M_X^{fin})$ onto $D(X)$.*

Proof. Given $p \in S[\mathfrak{g}^*]$, it defines at the same time a cohomology class $c(p) \in H_G^*(T_G^*M_X^{fin})$ (given by the Chern–Weil morphism) and also a differential operator with constant coefficients on \mathfrak{g} , we shall write it as $p(\partial)$. Now notice that the Poincaré duality pairing $([\gamma], c(p))$ is given by

$$\int_Q \gamma p(R) = \left(\int_Q \gamma p(R) e^{i\langle R, \xi \rangle} \right)_{|\xi=0} = (p(\partial) \text{infdex}(\gamma))_{|\xi=0}.$$

By Theorem 3.3 we have in our case that the Chern-Weil map $c : S[\mathfrak{g}^*] \rightarrow H_G^*(T_G^*M_X^{fin})$ is surjective with kernel I_X . From this and the previous considerations everything follows. \square

In order to proceed we need to recall a few definitions. For a rational subspace \underline{s} , we may consider the subspace $M_{\underline{s}} := \oplus_{a \in X \cap \underline{s}} L_a$ of M_X , we also denote by $G_{\underline{s}}$ the subgroup of G joint kernel of the elements in $X \cap \underline{s}$. The group $G_{\underline{s}}$ acts trivially on $M_{\underline{s}}$ inducing an action of $G/G_{\underline{s}}$.

Definition 5.13. We define the set $M_{\underline{s}}^f$ to be the open set of $M_{\underline{s}}$ where $G/G_{\underline{s}}$ acts with finite stabilizers.

Thus

$$H_{G,c}^*(M_{\underline{s}}^f) = S[\mathfrak{g}^*] \otimes_{S[(\mathfrak{g}/\mathfrak{g}_{\underline{s}})^*]} H_{G/G_{\underline{s}},c}^*(M_{\underline{s}}^f)$$

where \mathfrak{g}^* is in degree 2. In particular, by Proposition 5.10, we deduce that $H_{G,c}^{2i+1}(T_G^*M_{\underline{s}}^f) = 0$.

Now set $\tilde{T}_G^*M_{\underline{s}}^f := T_G^*M_X|_{M_{\underline{s}}^f}$, the restriction of $T_G^*M_X$ to $M_{\underline{s}}^f$. We see that $\tilde{T}_G^*M_{\underline{s}}^f = T_G^*M_{\underline{s}}^f \times M_{X \setminus \underline{s}}^*$, so we have a Thom isomorphism

$$C_{\underline{s}} : H_{G,c}^{2i}(T_G^*M_{\underline{s}}^f) \rightarrow H_{G,c}^{2(i+|X \setminus \underline{s}|)}(\tilde{T}_G^*M_{\underline{s}}^f), \quad H_{G,c}^{2i+1}(\tilde{T}_G^*M_{\underline{s}}^f) = 0.$$

Choose $0 \leq i \leq s$. We pass now to study the G -invariant open subspace $M_{X, \geq i}$ of M . The set $M_{X, \geq i+1}$ is open in $M_{X, \geq i}$ with complement the set $M_{=i}$, disjoint union of the sets $M_{\underline{s}}^f$ with $\underline{s} \in \mathcal{S}_X(i)$. Denote by $\tilde{T}_G^*M_{=i}$ the restriction of T_G^*M to $M_{=i}$, disjoint union of the sets $\tilde{T}_G^*M_{\underline{s}}^f$. Denote

$$j : M_{X, \geq i+1} \rightarrow M_{X, \geq i} \quad \text{the open inclusion}$$

$$e : \tilde{T}_G^*M_{=i} = \cup_{\underline{s}} \tilde{T}_G^*M_{\underline{s}}^f \rightarrow T_G^*M_{X, \geq i} \quad \text{the closed embedding.}$$

Let C_i be the Thom isomorphism from $H_{G,c}^{2i}(T_G^*M_{=i})$ to $H_{G,c}^{2(i+|X \setminus \underline{s}|)}(\tilde{T}_G^*M_{=i})$, the direct sum of the Thom isomorphisms $C_{\underline{s}}$.

Theorem 5.14. *For each $0 \leq i \leq s-1$,*

- a) For each $h \geq 0$, $H_{G,c}^{2h+1}(T_G^*M_{X,\geq i}) = 0$.*
- b) For each $h \geq 1$, the following sequence is exact*

(24)

$$0 \rightarrow H_{G,c}^{2h}(T_G^*M_{X,\geq i+1}) \xrightarrow{j^*} H_{G,c}^{2h}(T_G^*M_{X,\geq i}) \xrightarrow{C_i^{-1}e^*} \bigoplus_{\underline{s} \in \mathcal{S}_X(i)} H_{G,c}^{2h-2|X \setminus \underline{s}|}(T_G^*M_{\underline{s}}^f) \rightarrow 0.$$

Proof. Since $M_{X,\geq s} = M_X^{fin}$, we can assume by induction on $s-i$, that *a)* holds for each $j > i$. Also since $M_{=i}$ is the disjoint union of the spaces $M_{\underline{s}}^f$ which have no odd equivariant cohomology with compact supports, we get that $H_{G,c}^{2i+1}(T_G^*M_{=i}) = 0$ for each $0 \leq i \leq s-1$. Using this fact, both statements follow immediately from the long exact sequence of equivariant cohomology with compact supports associated to j, e . \square

Let us now make a simple but important remark.

Lemma 5.15. *Let $\underline{s} \in \mathcal{S}_X(j)$ with $j < k$. The element $d_{X \setminus \underline{s}} \in S[\mathfrak{g}^*]$ lies in the annihilator of $H_{G,c}^*(T_G^*M_{X,\geq k})$.*

Proof. $H_{G,c}^*(T_G^*M_{X,\geq k})$ is a module over $H_G^*(T_G^*M_{X,\geq k})$ and hence also over $H_G^*(M_{X,\geq k})$. Thus this lemma follows from Lemma 3.2. \square

Let us now split $X = A \cup B$ and $M_X = M_A \oplus M_B$. Then

$$T^*M_X = M_X \times M_X^* = T^*M_A \times T^*M_B = T^*M_A \times M_B^* \times M_B.$$

Consider the inclusions

$$T^*M_A \xrightarrow{s} T^*M_A \times M_B^* \xrightarrow{i} T^*M_A \times M_B^* \times M_B$$

Each is the zero section of a trivial bundle. The moment map for T^*M_X restrict to $T^*M_A \times M_B^*$ to the moment map of the factor T^*M_A . Hence setting $\tilde{T}_G^*M_A := T_G^*M_A \times M_B^*$ we obtain the inclusions

$$T_G^*M_A \xrightarrow{s} \tilde{T}_G^*M_A \xrightarrow{i} T_G^*M_X.$$

To the inclusion i we can apply Proposition 5.8 and to the inclusion s also Remark 5.9.

In particular, we get a Thom isomorphism

$$s! = C_{M_B^*} : H_{G,c}^*(T_G^*M_A) \rightarrow H_{G,c}^{*+2|B|}(\tilde{T}_G^*M_A) \cong H_{G,c}^{*+2|B|}(T_G^*M_A \times M_B^*)$$

and a homomorphism $i_! : H_{G,c}^*(\tilde{T}_G^*M_A) \rightarrow H_{G,c}^*(T_G^*M_X)$.

Proposition 5.16. *Take $\sigma \in H_{G,c}^*(T_G^*M_X)$, then $(i \circ s)_! C_{M_B^*}^{-1} i^*(\sigma) = d_B \sigma$.*

Proof. We first observe that $(i \circ s)_! C_{M_B^*}^{-1} = i_!$. So $(i \circ s)_! C_{M_B^*}^{-1} i^*(\sigma) = i_! i^*(\sigma)$ and the statement follows from Proposition 5.8. \square

Corollary 5.17. *Take $\sigma \in H_{G,c}^*(T_G^*M_X)$. Let $\sigma_0 = C_{M_B^*}^{-1} i^*(\sigma) \in H_{G,c}^*(T_G^*M_A)$. Then, we have the equality of distributions:*

$$\partial_B(\text{infdex}(\sigma)) = \text{infdex}(\sigma_0).$$

Proof. We use the fact that the infinitesimal index commutes with $i_!$ (see [8], Theorem 4.9) and is a map of $S[\mathfrak{g}^*]$ modules. \square

We have defined in Definition 4.4 the space of distributions $\mathcal{G}(X)$ as those distributions f on \mathfrak{g}^* such that $\partial_{X \setminus \underline{r}} f \in \mathcal{S}'(\mathfrak{g}^*, \underline{r})$ for all $\underline{r} \in \mathcal{S}_X$ and $\tilde{\mathcal{G}}(X)$ as the $S[\mathfrak{g}^*]$ module generated by $\mathcal{G}(X)$. Then $\tilde{\mathcal{G}}_i(X)$ is the subspace in $\tilde{\mathcal{G}}(X)$ such that $\partial_{X \setminus \underline{r}} f = 0$ for all $\underline{r} \in \mathcal{S}_X(i-1)$.

Lemma 5.18. *For each $i \geq 0$, infdex maps $H_{G,c}^*(T_G^* M_{X, \geq i})$ to the space $\tilde{\mathcal{G}}_i(X)$.*

Proof. Denote by $\ell : \tilde{\mathcal{G}}_i(X) \rightarrow \tilde{\mathcal{G}}(X)$ the inclusion. By Lemma 5.15, if $\sigma \in H_{G,c}^*(T_G^* M_{X, \geq i})$ and \underline{r} is a rational subspace of dimension strictly less than i , we have $d_{X \setminus \underline{r}} \sigma = 0$. Thus $\partial_{X \setminus \underline{r}} \text{infdex}(\sigma) = 0$. It follows that the only thing we have to show is that, if $\sigma \in H_{G,c}^*(T_G^* M_X)$, then $\text{infdex}(\sigma)$ lies in $\tilde{\mathcal{G}}(X)$. Take a rational subspace \underline{s} . By Corollary 5.17, the infinitesimal index of $d_{X \setminus \underline{s}} \sigma$ equals the infinitesimal index of an element $\sigma_0 \in H_{G,c}^*(T_G^* M_{X \cap \underline{s}})$. But the action of G on $M_{X \cap \underline{s}}$ factors through the quotient $G/G_{\underline{s}}$ whose Lie algebra is $\mathfrak{g}/\mathfrak{g}_{\underline{s}}$. Thus $H_{G,c}^*(T_G^* M_{X \cap \underline{s}}) \cong S[\mathfrak{g}^*] \otimes_{S[(\mathfrak{g}/\mathfrak{g}_{\underline{s}})^*]} H_{G/G_{\underline{s}},c}^*(T_{G/G_{\underline{s}}}^* M_{X \cap \underline{s}})$, hence $\partial_{X \setminus \underline{s}} \text{infdex}(\sigma) \in S[\mathfrak{g}^*] \text{infdex}(H_{G/G_{\underline{s}},c}^*(T_{G/G_{\underline{s}}}^* M_{X \cap \underline{s}}))$.

But $\text{infdex}(H_{G/G_{\underline{s}},c}^*(T_{G/G_{\underline{s}}}^* M_{X \cap \underline{s}})) \subset \mathcal{S}'(\mathfrak{g}^*, \underline{r})$ hence the claim. \square

The following theorem characterizes the values of the infinitesimal index on the entire M_X . This time, we use the notations and the exact sequences contained in Theorem 5.14 Remark 4.11 and Corollary 5.17

Theorem 5.19. *For each $0 \leq i \leq s$,*

- *the diagram*

$$\begin{array}{ccccc}
 0 \rightarrow H_{G,c}^*(T_G^* M_{X, \geq i+1}) & \xrightarrow{j_*} & H_{G,c}^*(T_G^* M_{X, \geq i}) & \xrightarrow{C_i^{-1} e^*} & H_{G,c}^*(T_G^* M_{=i}) \rightarrow 0 \\
 \text{infdex} \downarrow & & \text{infdex} \downarrow & & \text{infdex} \downarrow \\
 0 \rightarrow \tilde{\mathcal{G}}_{i+1}(X) & \xrightarrow{\ell} & \tilde{\mathcal{G}}_i(X) & \xrightarrow{\mu_i} & \bigoplus_{\underline{s} \in \mathcal{S}_X(i)} D^{\mathfrak{g}}(X \cap \underline{s}) \rightarrow 0
 \end{array}$$

commutes.

- *Its vertical arrows are isomorphisms.*
- *In particular, the infinitesimal index gives an isomorphism between $H_{G,c}^*(T_G^* M_X)$ and $\tilde{\mathcal{G}}(X)$.*

Proof. Lemma 5.18 tells us that the diagram is well defined. We need to prove commutativity.

We prove that the square on the right hand side is commutative using Corollary 5.17. The square on the left hand side is commutative since j_* is compatible with the infinitesimal index and ℓ is the inclusion.

Recall that $H_{G,c}^*(T_G^* M_{X \cap \underline{s}}) \cong S[\mathfrak{g}^*] \otimes_{S[(\mathfrak{g}/\mathfrak{g}_{\underline{s}})^*]} H_{G/G_{\underline{s}},c}^*(T_{G/G_{\underline{s}}}^* M_{X \cap \underline{s}})$ and that

$$D^{\mathfrak{g}}(X \cap \underline{s}) = S[\mathfrak{g}^*] D(X \cap \underline{s}) \cong S[\mathfrak{g}^*] \otimes_{S[(\mathfrak{g}/\mathfrak{g}_{\underline{s}})^*]} D(X \cap \underline{s}).$$

Using Theorem 5.12, this implies that the right vertical arrow is always an isomorphism.

We want to apply descending induction on i . When $i + 1 = s$, since $M_{X, \geq s} = M_X^{fin}$ and $\tilde{\mathcal{G}}_{s-1}(X) = D(X)$, Theorem 5.12 gives that the left vertical arrow is an isomorphism. So assume that the left vertical arrow is an isomorphism. We then deduce from the five Lemma that the central vertical arrow is an isomorphism and conclude. \square

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